Exercise 8F

1 a $y = x^2 - 7x + 10$

$\frac{dy}{dx} = 2x - 7$

When $x = 2$, gradient $= 2 \times 2 - 7 = -3$

So the equation of the tangent at (2, 0) is

$y - 0 = -3(x - 2)$

$y = -3x + 6$

$y + 3x - 6 = 0$

b $y = x + \frac{1}{x} = x + x^{-1}$

$\frac{dy}{dx} = 1 - x^{-2}$

When $x = 2$, gradient $= 1 - 2^{-2} = \frac{3}{4}$

So the equation of the tangent at $\left(2, 2\frac{1}{2}\right)$ is

$y - 2\frac{1}{2} = \frac{3}{4}(x - 2)$

$4y - 10 = 3x - 6$

$4y - 3x - 4 = 0$

c $y = 4\sqrt{x} = 4x^{\frac{1}{2}}$

$\frac{dy}{dx} = 2x^{-\frac{1}{2}}$

When $x = 9$, gradient $= 2 \times 9^{-\frac{1}{2}} = \frac{2}{3}$

So the equation of the tangent at (9, 12) is

$y - 12 = \frac{2}{3}(x - 9)$

$3y - 36 = 2x - 18$

$3y - 2x - 18 = 0$

d $y = \frac{2x - 1}{x} = \frac{2x}{x} - \frac{1}{x} = 2 - x^{-1}$

$\frac{dy}{dx} = 0 + x^{-2} = x^{-2}$

When $x = 1$, gradient $= 1^{-2} = 1$

So the equation of the tangent at (1, 1) is

$y - 1 = 1 \times (x - 1)$

$y = x$

e $y = 2x^3 + 6x + 10$

$\frac{dy}{dx} = 6x^2 + 6$

When $x = -1$, gradient $= 6(-1)^2 + 6 = 12$

e So the equation of the tangent at $(-1, 2)$ is

$y - 2 = 12(x - (-1))$

$y - 2 = 12x + 12$

$y = 12x + 14$

f $y = x^2 - \frac{7}{x^2} = x^2 - 7x^{-2}$

$\frac{dy}{dx} = 2x + 14x^{-3}$

When $x = 1$, gradient $= 2 + 14 = 16$

So the equation of the tangent at $(1, -6)$ is

$y - (-6) = 16(x - 1)$

$y + 6 = 16x - 16$

$y = 16x - 22$

2 a $y = x^2 - 5x$

$\frac{dy}{dx} = 2x - 5$

When $x = 6$, gradient of curve $= 2 \times 6 - 5$
$= 7$

So gradient of normal is $-\frac{1}{7}$.

The equation of the normal at (6, 6) is

$y - 6 = -\frac{1}{7}(x - 6)$

$7y - 42 = -x + 6$

$7y + x - 48 = 0$

b $y = x^2 - \frac{8}{\sqrt{x}} = x^2 - 8x^{-\frac{1}{2}}$

$\frac{dy}{dx} = 2x + 4x^{-\frac{3}{2}}$

When $x = 4$, gradient of curve

$= 2 \times 4 + 4(4)^{-\frac{3}{2}} = 8 + \frac{4}{8} = \frac{17}{2}$

So gradient of normal is $-\frac{2}{17}$.

The equation of the normal at (4, 12) is

$y - 12 = -\frac{2}{17}(x - 4)$

$17y - 204 = -2x + 8$

$17y + 2x - 212 = 0$

3 $y = x^2 + 1$

$\frac{dy}{dx} = 2x$

When $x = 2$, $\frac{dy}{dx} = 4$

So the equation of the tangent at (2, 5) is

$y - 5 = 4(x - 2)$

$y = 4x - 3$

When $x = 1$, gradient of curve = 2

So gradient of normal is $-\frac{1}{2}$.

The equation of the normal is

$y - 2 = -\frac{1}{2}(x - 1)$

$y = -\frac{1}{2}x + 2\frac{1}{2}$

Tangent at (2, 5) and normal at (1, 2) meet when

$4x - 3 = -\frac{1}{2}x + 2\frac{1}{2}$

$8x - 6 = -x + 5$

$9x = 11$

$x = \frac{11}{9}$

$y = 4 \times \frac{11}{9} - 3 = \frac{17}{9}$

So the tangent at (2, 5) meets the normal at (1, 2) at $\left(\frac{11}{9}, \frac{17}{9}\right)$.

4 $y = x + x^3$

$\frac{dy}{dx} = 1 + 3x^2$

When $x = 0$, gradient of curve $= 1 + 3 \times 0^2$
$= 1$

So gradient of normal is $-\frac{1}{1} = -1$.

The equation of the normal at (0, 0) is

$y - 0 = -1(x - 0)$

$y = -x$

When $x = 1$, gradient of curve $= 1 + 3 \times 1^2$
$= 4$

So gradient of normal is $-\frac{1}{4}$.

The equation of the normal at (1, 2) is

$y - 2 = -\frac{1}{4}(x - 1)$

$4y - 8 = -x + 1$

$4y + x - 9 = 0$

Normals at (0, 0) and (1, 2) meet when

$4(-x) + x - 9 = 0$

$3x = -9$

$x = -3$

$y = 3$

The normals meet at (−3, 3).

5 $y = f(x) = 12 - 4x + 2x^2$

$\frac{dy}{dx} = 0 - 4 + 4x$

When $x = -1$, $y = 12 - 4(-1) + 2(-1)^2$
$= 18$

$\frac{dy}{dx} = 4(1) = -4$

The tangent at (−1, 18) has gradient −4.

So its equation is

$y - 18 = -4(x + 1)$

$y - 18 = -4x - 4$

$y = 14 - 4x$

The normal at (−1, 18) has gradient $\frac{-1}{-4} = \frac{1}{4}$. So its equation is

$y - 18 = \frac{1}{4}(x + 1)$

$4y - 72 = x + 1$

$4y - x - 73 = 0$

6 $y = 2x^2$

$\frac{dy}{dx} = 4x$

When $x = \frac{1}{2}$, $y = 2 \times \left(\frac{1}{2}\right)^2 = \frac{1}{2}$

$\frac{dy}{dx} = 4 \times \frac{1}{2} = 2$

So gradient of normal is $-\frac{1}{2}$.

The equation of the normal at $\left(\frac{1}{2}, \frac{1}{2}\right)$ is

$y - \frac{1}{2} = -\frac{1}{2}\left(x - \frac{1}{2}\right)$

$y = -\frac{1}{2}x + \frac{3}{4}$

6 The normal intersects the curve when

$$2x^2 = -\frac{1}{2}x + \frac{3}{4}$$
$$8x^2 + 2x - 3 = 0$$
$$(4x + 3)(2x - 1) = 0$$
$$x = -\frac{3}{4} \text{ or } \frac{1}{2}$$

$x = \frac{1}{2}$ is point P,

so $x = -\frac{3}{4}$ must be point Q.

When $x = -\frac{3}{4}$, $y = -\frac{1}{2}\left(-\frac{3}{4}\right) + \frac{3}{4} = \frac{9}{8}$

Point Q is $\left(-\frac{3}{4}, \frac{9}{8}\right)$.

Challenge

$y = 4x^2 + 1$

$\frac{dy}{dx} = 8x$

Gradient of line $L = 8x$

Equation of line L:

$y = 8x(x) + c$
$= 8x^2 + c$

Line L passes through the point $(0, -8)$,
so $c = -8$

$y = 8x^2 - 8$

Line L meets the curve when

$$4x^2 + 1 = 8x^2 - 8$$
$$4x^2 = 9$$
$$x^2 = \frac{9}{4}$$
$$x = \pm\frac{3}{2}$$

As the gradient is positive, $x = \frac{3}{2}$

$$y = 8x(x) - 8$$
$$= 8\left(\frac{3}{2}\right)x - 8$$
$$= 12x - 8$$